The largest linear space of operators satisfying the Daugavet Equation in L_1

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Abstract

We find the largest linear space of bounded linear operators on $L_1(\Omega)$ that being restricted to any $L_1(A)$, $A \subset \Omega$, satisfy the Daugavet equation.

1 Introduction.

Let (Ω, Σ, μ) be an arbitrary measure space without atoms of infinite measure. Let also $\Sigma^+ = \{A \in \Sigma : \mu(A) > 0\}$. If $A \in \Sigma^+$, $L_1(A)$ stands for the space of (classes of) μ-integrable functions supported on A. If T is a bounded linear operator on $L_1(\Omega)$ and $A \in \Sigma^+$, we denote by T_A the restriction of T onto $L_1(A)$. Finally, $\mathcal{L}(L_1(\Omega))$ denotes the space of all bounded linear operators on $L_1(\Omega)$.

The purpose of this note is to give an explicit description of the largest linear space $\mathcal{M}$ of operators $T \in \mathcal{L}(L_1(\Omega))$ satisfying the following identity:

$$\|Id_A + T_A\| = 1 + \|T_A\|, \tag{1}$$

for any set $A \in \Sigma^+$.

Identity (1) is known as the Daugavet equation and is investigated in a series of works (see [4] and [6] for recent results and further references). It was first discovered by Babenko and Pichugov ([1]) that all the compact operators on $L_1[0,1]$ satisfy (1), if $A = [0,1]$. Later, Holub proved the same result for the weakly compact operators on an arbitrary atomless $L_1(\Omega)$ (see [3]). Plichko and Popov in their work [5] found much broader (in case of atomless μ) linear class of so-called narrow operators satisfying the Daugavet equation, and in fact their proof works for operators from $L_1(A)$ to $L_1(\Omega)$, whenever $A \in \Sigma^+$.

So, finding the largest class of such operators naturally completes this line of results.

2 Main result.

In the sequel it is convenient to denote $\Sigma_A^+ = \{B : B \subset A,\ B \in \Sigma^+\}$, whenever $A \in \Sigma^+$.

We define $\mathcal{M}$ as the set of all operators $T \in \mathcal{L}(L_1(\Omega))$ that meet the following condition:

$$\text{for every } \varepsilon > 0 \text{ and } A \in \Sigma^+ \text{ there is a } B \in \Sigma_A^+ \text{ with } \mu(B) < \infty \text{ such that } \left\| \chi_B \cdot T\left(\frac{\chi_B}{\mu(B)}\right) \right\| < \varepsilon. \tag{2}$$

This condition simply means that the operator T can shift sufficiently many functions from their supports.

Let us state our main result.

Theorem 1 *Every linear set of operators satisfying (1) for any $A \in \Sigma^+$ is contained in $\mathcal{M}$, and $\mathcal{M}$ is itself a closed linear space consisting of such operators.*

The main ingredient in the proof of this theorem is the following proposition.

Proposition 2 *For an operator $T \in \mathcal{L}(L_1(\Omega))$ the following conditions are equivalent:*

(i) T and $-T$ satisfy (1) for all $A \in \Sigma^+$;

(ii) *For every $\varepsilon > 0$ and $A \in \Sigma^+$ there is an $A' \in \Sigma_A^+$ such that if $B \in \Sigma_{A'}^+$ then we can find a $B' \in \Sigma_B^+$ with the following properties:*

a) $\left\| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} \right\| < \varepsilon$*,*

b) $\left\| \chi_{B'} \cdot T\left(\frac{\chi_{B'}}{\mu(B')}\right) \right\| < \varepsilon$*;*

(iii) $T \in \mathcal{M}$*.*

Proof. (i) implies (ii). We begin with the following observation.

Suppose $S : L_1(A) \mapsto L_1(\Omega)$ is a bounded linear operator, then any given $\varepsilon > 0$ there is a set $A_1 \in \Sigma_A^+$ with $\mu(A_1) < \infty$ such that for every non-negative function $f \in S(L_1(A_1))$ we have $\|Sf\| > \|S\| - \varepsilon$.

Indeed, we can assume that $\mu(A) < \infty$ and choose $g^* \in S(L_1^*(\Omega))$ so that $\|S^* g^*\| > \|S\| - \varepsilon$. Then, regarding $S^* g^*$ as an element of $L_\infty(A)$ we find a set $A_1 \in \Sigma_A^+$ with $\theta S^* g^*(A_1) \subset (\|S\| - \varepsilon, \|S\|]$, where θ is a sign. Now, if $f \in S(L_1(A))$, $f \geq 0$ and $\operatorname{supp}(f) \subset A_1$, then $\|Sf\| > \theta g^*(Sf) = \theta S^* g^*(f) > \|S\| - \varepsilon$, from where the observation follows.

We know that $\|Id_A + T_A\| = 1 + \|T_A\|$. By scaling, without loss of generality we can and do assume that $\|T_A\| = 1$. So there is an $A_1 \in \Sigma_A^+$ with $\mu(A_1) < \infty$ such that

$$\left\| \frac{\chi_B}{\mu(B)} + T\left(\frac{\chi_B}{\mu(B)}\right) \right\| > 2 - \varepsilon, \tag{3}$$

whenever $B \in \Sigma_{A_1}^+$. We also know that $\|Id_{A_1} - T_{A_1}\| = 1 + \|T_{A_1}\| > 2 - \varepsilon$. Thus there exists an $A' \in \Sigma_{A_1}^+$ such that

$$\left\| \frac{\chi_B}{\mu(B)} - T\left(\frac{\chi_B}{\mu(B)}\right) \right\| > 2 - \varepsilon, \tag{4}$$

whenever $B \in \Sigma_{A'}^+$.

We prove that A' is the desired set.

To this end, let us fix $B \in \Sigma_{A'}^+$. It follows from (3), (4) and a theorem of Dor [2] that there are two disjoint measurable sets Ω_1 and Ω_2 in Ω such that

$$\int_{\Omega_1} \left| T\left(\frac{\chi_B}{\mu(B)}\right) \right| (t)dt > (1 - \varepsilon)^2, \tag{5}$$

and

$$\int_{\Omega_2} \frac{\chi_B}{\mu(B)}(t)dt > (1 - \varepsilon)^2.$$

The last inequality implies

$$\begin{aligned}\mu(B \cap \Omega_1) &= \mu(B) \int_{B \cap \Omega_1} \frac{\chi_B}{\mu(B)}(t)dt < \mu(B) \int_{\Omega \setminus \Omega_2} \frac{\chi_B}{\mu(B)}(t)dt \\ &< (1-(1-\varepsilon)^2)\mu(B) = (2\varepsilon - \varepsilon^2)\mu(B).\end{aligned} \tag{6}$$

Let us put $B' = B \setminus \Omega_1$ and show that B' meets conditions a) and b).

First,

$$\begin{aligned}\left\| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} \right\| &= \int_\Omega \left| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_{B'}}{\mu(B)} + \frac{\chi_{B'}}{\mu(B)} - \frac{\chi_B}{\mu(B)} \right| (t)dt \\ &\leq 1 - \frac{\mu(B')}{\mu(B)} + \frac{\mu(B \cap \Omega_1)}{\mu(B)} = 2\frac{\mu(B \cap \Omega_1)}{\mu(B)} \qquad ,\end{aligned}$$

and taking into account (6), we obtain

$$\left\| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} \right\| < 2(2\varepsilon - \varepsilon^2). \tag{7}$$

Second, from (5), (7) and $\|T_A\| = 1$ it follows that

$$\begin{aligned}\left\| \chi_{B'} \cdot T\left(\frac{\chi_{B'}}{\mu(B')}\right) \right\| &= \int_{B'} \left| T\left(\frac{\chi_{B'}}{\mu(B')}\right) \right| (t)dt \\ &< \int_{B'} \left| T\left(\frac{\chi_B}{\mu(B)}\right) \right| (t)dt + 2(2\varepsilon - \varepsilon^2) \\ &\leq \int_{\Omega \setminus \Omega_1} \left| T\left(\frac{\chi_B}{\mu(B)}\right) \right| (t)dt + 2(2\varepsilon - \varepsilon^2) \\ &\leq 3(2\varepsilon - \varepsilon^2).\end{aligned}$$

In view of arbitrariness of ε, this gives the desired result.

It is obvious that (iii) follows from (ii).

Let us finally prove that (iii) implies (i). Since $\mathcal{M}$ is stable under scalar multiplication, it is sufficient to prove (1) only for T.

To this end, we fix an arbitrary $A \in \Sigma^+$ and as above for any given $\varepsilon > 0$ we find an $A' \in \Sigma_A^+$ with $\mu(A') < \infty$ such that for every $B \in \Sigma_{A'}^+$, $\left\| T\left(\frac{\chi_B}{\mu(B)}\right) \right\| > \|T_A\| - \varepsilon$. By condition (2), there is a $B_0 \in \Sigma_{A'}^+$ such that $\left\| \chi_{B_0} \cdot T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right\| < \varepsilon$. This means that $\frac{\chi_{B_0}}{\mu(B_0)}$ and $T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right)$ are almost disjoint functions, and as a consequence we have the following estimate:

$$
\begin{aligned}
\|Id_A + T_A\| &\geq \left\| \frac{\chi_{B_0}}{\mu(B_0)} + T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right\| \\
&= \int_{B_0} \left| \frac{\chi_{B_0}}{\mu(B_0)} + T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right| (t)dt + \int_{\Omega} \left| T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right| (t)dt \\
&\quad - \int_{B_0} \left| T\left(\frac{\chi_{B_0}}{\mu(B_0)}\right) \right| (t)dt \\
&> 1 - \varepsilon + \|T_A\| - \varepsilon - \varepsilon = 1 + \|T_A\| - 3\varepsilon.
\end{aligned}
$$

This finishes the proof. □

Now we are in a position to prove our main result.

Proof of Theorem 1.

Proposition 2 easily implies that $\mathcal{M}$ is largest and consists of operators satisfying (1) for all $A \in \Sigma^+$. $\mathcal{M}$ is obviously closed and stable under scaling. So, the only thing we have to prove is that if operators U and V belong to $\mathcal{M}$, then their sum belong to $\mathcal{M}$ too. To show this, we check condition (ii) of Proposition 2 for $U + V$. Further on, we assume that $\|V\| \leq 1$.

Indeed, let $A \in \Sigma^+$ and $\varepsilon > 0$ be arbitrary. Applying Proposition 2 to the operator U we find a set $A' \in \Sigma_A^+$ as in condition (ii). Then, by the same proposition applied to V we find a set $A'' \in \Sigma_{A'}^+$ with the correspondent properties. To show that A'' is the required set, suppose $B \in \Sigma_{A''}^+$. By the choice of A'' there is a $B' \in \Sigma_B^+$ such that

$$\left\| \frac{\chi_{B'}}{\mu(B')} - \frac{\chi_B}{\mu(B)} \right\| < \frac{\varepsilon}{4}, \tag{8}$$

and

$$\left\| \chi_{B'} \cdot V\left(\frac{\chi_{B'}}{\mu(B')}\right) \right\| < \frac{\varepsilon}{4}. \tag{9}$$

Since $B' \subset A'$, by the analogous property of A', there is a $B'' \in \Sigma_{B'}^+$ with

$$\left\| \frac{\chi_{B''}}{\mu(B'')} - \frac{\chi_{B'}}{\mu(B')} \right\| < \frac{\varepsilon}{4}, \tag{10}$$

and

$$\left\| \chi_{B''} \cdot U\left(\frac{\chi_{B''}}{\mu(B'')}\right) \right\| < \frac{\varepsilon}{2}.$$

From (8) and (10) we get $\left\| \frac{\chi_{B''}}{\mu(B'')} - \frac{\chi_B}{\mu(B)} \right\| < \varepsilon$. So, if we prove that

$$\left\| \chi_{B''} \cdot V \left(\frac{\chi_{B''}}{\mu(B'')} \right) \right\| < \frac{\varepsilon}{2},$$

then

$$\left\| \chi_{B''} \cdot (V + U) \left(\frac{\chi_{B''}}{\mu(B'')} \right) \right\| < \varepsilon,$$

and we are done. But this easily follows from (9), (10) and the facts that $\|V\| \le 1$ and $B'' \subset B'$.

The proof is completed. $\square$

References

[1] V. F. Babenko and S. A. Pichugov, On a property of compact operators in the space of integrable functions, *Ukranian Math. J.* **33** (1981), 374-376.

[2] L. E. Dor, On projections in L_1, *Ann. of Math.* **102** (1975), 463-474.

[3] J. R. Holub, Daugavet's equation and operators on $L_1(\mu)$, *Proc. Amer. Math. Soc.* **100** (1987), 295–300.

[4] V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner, Banach spaces with the Daugavet property, *C.R.Acad.Sci.Paris* **325.1** (1997), 1291-1294. (to appear in Trans. Amer. Math. Soc.)

[5] A. M. Plichko and M. M. Popov, Symmetric function spaces on atomless probability spaces, *Rozpr.Mat.* **306** (1990), 1-85.

[6] R. V. Shvidkoy, Geometric aspects of the Daugavet property, (to appear).